

On the Dirac Eigenvalues as Observables of the on-shell $N = 2$ $D = 4$ Euclidean Supergravity

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Abstract

We generalize previous works on the Dirac eigenvalues as dynamical variables of the Euclidean gravity and $N = 1$ $D = 4$ supergravity to on-shell $N = 2$ $D = 4$ Euclidean supergravity. The covariant phase space of the theory is defined as the space of the solutions of the equations of motion modulo the on-shell gauge transformations. In this space we define the Poisson brackets and compute their value for the Dirac eigenvalues.

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1 Introduction

One of the most important and longstanding problems in the general relativity and its supersymmetric extensions is finding a complete set of observables of these theories. In particular, such objects should contain information about spacetime manifold and have vanishing Poisson brackets with the constraints of the general relativity. Therefore, they might be useful for quantizing the gravity and the supergravity by employing the canonical methods [1, 2]. Several years ago, it was shown in the framework of the noncommutative geometry that there is a relationship between the algebra generated by the Dirac operator and the smooth functions on a manifold and the geometry of the latter [3]. On the other hand, it is known from the spectral geometry that the eigenvalues of certain differential operators contain information on the geometry of the manifold on which they are defined [4]. These observations were applied to the general relativity in [5] where it was proposed that the eigenvalues of the Weyl operator played the sought for role in the Euclidean general relativity in the Ashtekar variables. However, there are two major problems in making this idea work: the difficulty of calculating the spectrum of the Weyl operator and the uniqueness of the gauge variations of theory in terms of the Ashtekar variables. Since the theory of the Dirac operator is better known, it has been put forward in [6, 7, 8] that the eigenvalues of the Dirac operator be considered as dynamical variables of the gravitational field in the Euclidean general relativity. The theory is restricted to the compact Euclidean manifolds since the Dirac operator should be hermitean and elliptic. Also, it is precisely in the Euclidean case that one can make the connection with the noncommutative geometry. In particular, the Hilbert-Einstein action of the general relativity can be written solely in terms of the eigenvalues of the Dirac operator as a Dixmier trace [6] (see for noncommutative geometry, e. g. [3, 9] and for connection with spaces with torsion [10, 11, 12, 13, 14, 15].)

In a previous work, the above construction was extended to the case in which, beside the diffeomorphisms and $SO(4)$ rotations, local $N = 1$ supersymmetry transformations have been taken into account [16]. The relations resulting from the variation of the Dirac eigenvalues were considered *a priori* as constraints on the geometry of the Euclidean manifolds and some consequences of this hypothesis were analyzed in [17, 18, 19, 20] (see also the review in [21]). In this article, we are going to consider the on-shell $N = 2$ $D = 4$ Euclidean supergravity. The reasons for which one would like to understand better the Euclidean supergravity come mainly from the AdS/CFT correspondence in string theory (see [22] and the references therein) and the study of instantons in the field theories coupled with gravity.

In the Euclidean theory, the complex Dirac spinor can be written as two Majorana spinors defined by a new Majorana condition and the bosonic fields are real [23, 24, 25]. This allows us to write local supersymmetry transformations consistent with $SO(4)$ as in [26]¹. Our main aim is to extend the formulation of the $N = 1$ $D = 4$ Euclidean supergravity in which the dynamical variables are given by the Dirac eigenvalues λ_n 's to the on-shell $N = 2$ $D = 4$ Euclidean supergravity. To this end, one has to make use of the concept of the covariant phase space of supergravity. The key objects defined on it are the covariant Poisson brackets. We define the Poisson brackets for the $N = 2$ $D = 4$ Euclidean supergravity and derive their expression for the Dirac eigenvalues.

The plan of this paper is as follows. In the Section 2 we discuss the on-shell $N = 2$ $D = 4$ Euclidean supergravity in terms of the Dirac eigenvalues. In Section 3 we define the Poisson brackets in the covariant phase space defined as the space of the solutions of the equations of motion modulo the local gauge transformations. The Poisson brackets of the

¹For different approaches to the Euclidean supersymmetry see the review in [24] and the references therein.

Dirac eigenvalues are computed in the Section 4. The last section is devoted to conclusions. The units in this paper are such that $32\pi G = 1$.

2 Dirac Eigenvalues as Dynamical Variables of $N = 2$ $D = 4$ Euclidean Supergravity

Consider the $N = 2$ $D = 4$ Euclidean supergravity on a compact spacetime manifold M without boundary. The gravitational supermultiplet has the following field content: the real vielbein e_μ^a , the real vector field A_μ and the gravitino ψ_μ^α . The Euclidean spinors are obtained from the complex Majorana spinor $\psi_\mu^{M\alpha}$ in the Minkowski spacetime by performing the following Wick rotation [23, 24]

$$\psi^M \rightarrow O\psi, \quad \psi^{M\dagger} \rightarrow \psi^\dagger O, \quad O = e^{\gamma^4 \gamma^5 \pi/4}. \quad (1)$$

The matrix O defined above acts on the Dirac matrices γ^M in the Minkowski spacetime and transforms them into matrices in the Euclidean spacetime according to the rule

$$\gamma^{Ma} \rightarrow O^{-1} \gamma^a O. \quad (2)$$

The action of the $N = 2$ $D = 4$ Euclidean supergravity takes the following form [26]

$$S = \int d^4x \left[-\frac{e}{2\kappa^2} R(e, \omega) - \varepsilon^{\mu\nu\rho\sigma} \psi_\mu^\dagger \gamma_\nu \nabla_\rho \psi_\sigma + \frac{1}{4} e F^2 - \frac{6g^2}{\kappa^4} e \right. \quad (3)$$

$$\left. - \frac{i\kappa}{2\sqrt{2}} \psi_\mu^\dagger \gamma_5 \left(e (F^{\mu\nu} + \tilde{F}^{\mu\nu}) + \gamma_5 (\tilde{F}^{\mu\nu} + \tilde{F}^{\mu\nu}) + \frac{4ig}{\kappa^2} e \gamma^{\mu\nu} \right) \psi_\nu \right], \quad (4)$$

where

$$F^{\mu\nu} = F^{\mu\nu} - \frac{i\kappa}{\sqrt{2}} (\psi_\mu^\dagger \gamma_5 \psi_\nu - \psi_\nu^\dagger \gamma_5 \psi_\mu), \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \tilde{F}_{\rho\sigma}, \quad (5)$$

$$\omega_{\mu ab} = \omega_{\mu ab}(e) - \frac{\kappa^2}{4} (\psi_a^\dagger \gamma_5 \gamma_\mu \psi_b + \psi_a^\dagger \gamma_5 \gamma_b \psi_\mu - \psi_a^\dagger \gamma_5 \gamma_b \psi_a - (a \longleftrightarrow b)), \quad (6)$$

are the supercovariant field strength, its Hodge dual and the spin connection, respectively. The phase space of the theory is a (super)covariant notion and it is defined as the space of the solutions of the equations of motion (e, A, ψ) modulo the gauge transformations. Thus, it is sufficient to consider only the on-shell supergravity transformations [26]

$$\begin{aligned} \delta e_\mu^a &= \xi^\nu \partial_\nu e_\mu^a + \partial_\mu \xi^\nu e_\nu^a - \xi_b^a e_\mu^b + \frac{1}{2} (\psi_\mu^\dagger \gamma_5 \gamma^a \epsilon - \epsilon^\dagger \gamma_5 \gamma^a \psi_\mu) \\ \delta A_\mu &= \xi^\nu \partial_\nu A_\mu + \partial_\mu \xi^\nu A_\nu + \partial_\mu \xi + \frac{i}{\sqrt{2}} (\epsilon^\dagger \gamma_5 \psi_\mu - \psi_\mu^\dagger \gamma_5 \epsilon) \\ \delta \psi_\mu &= \xi^\nu \partial_\nu \psi_\mu + \partial_\mu \xi^\nu \psi_\nu + \frac{1}{4} \xi^{ab} \gamma_{ab} \psi_\mu - \epsilon^\dagger \nabla_\mu \epsilon + \frac{i}{2\sqrt{2}} \gamma^\nu G_{\mu\nu} \epsilon \\ &\quad - \frac{1}{\sqrt{2}} g^{-2} \gamma_\mu \epsilon + i g \xi \psi_\mu \\ \delta \psi_\mu^\dagger &= \xi^\nu \partial_\nu \psi_\mu^\dagger + \partial_\mu \xi^\nu \psi_\nu^\dagger - \frac{1}{4} \xi^{ab} \psi_\mu^\dagger \gamma_{ab} + \epsilon^\dagger \nabla_\mu \epsilon + \frac{i}{2\sqrt{2}} \epsilon^\dagger G_{\mu\nu} \gamma^\nu \\ &\quad + \frac{1}{\sqrt{2}} g^{-2} \epsilon^\dagger \gamma_\mu - i g \xi \psi_\mu^\dagger. \end{aligned} \quad (7)$$

Here, the parameters of the general coordinate transformations, local $SO(4)$ rotations, $U(1)$ gauge transformations and $N = 2$ Euclidean supersymmetry transformations are denoted by ξ^μ, ξ^{ab}, ξ and $\epsilon, \epsilon^\dagger$, respectively.

The observables are functions on the phase space. Let us denote by \mathcal{F} the space of smooth supergravity multiplets (e, A, ψ) which are composed by smooth tetrads, smooth connections on the $U(1)$ bundle over M and smooth sections from the spin bundle $\Sigma(M)$. The space of solutions of the equations of motion of the Euclidean supergravity is denoted by \mathcal{S} and the space of the gauge orbits of the transformations (7) in \mathcal{F} is denoted by \mathcal{O} . Then the covariant phase space is \mathcal{G} .

As was noted in [6, 16] a set of classical observables of the Euclidean gravity and $N = 1$ Euclidean supergravity on compact manifolds is given by the eigenvalues $\{\lambda_n\}$'s of the Dirac operator which define a discrete family of real-valued functions on the space of vielbeins. The functions $\lambda_n(e, A, \psi)$ are invariant under the gauge transformations for each n . Let us now apply the same idea to the $N = 2$ $D = 4$ Euclidean supergravity with the on-shell transformations given by the relations (7). The Dirac operator has the following form

$$\mathcal{D} = i\gamma^a e_a^\mu \nabla_\mu. \quad (8)$$

and it is given by the sum between the Dirac operator in the non-supersymmetric theory and two terms depending on the potential A and the gravitino ψ , respectively. The gauge- and Lorentz-covariant derivative of the $N = 2$ $D = 4$ Euclidean are obtained from the corresponding covariant derivative of the $N = 2$ supergravity in the Minkowski background by a Wick rotation [23, 24]

$$\nabla_\mu \psi_\nu = (D_\mu - igA_\mu) \psi_\nu, \quad (9)$$

where the operator D_μ has the following action on the vielbein and the gravitino fields

$$D_\mu e_\mu^a = \left(\partial_\mu - \omega_{\mu b}^a \right) e_\nu^b, \quad (10)$$

$$D_\mu \psi_\mu = \left(\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \right) \psi_\nu. \quad (11)$$

If one fixes the A and ψ fields, the symbol of the Dirac operator in the presence of supersymmetry is a local isomorphism

$$L_0 : U \rightarrow \text{Hom}(S(M), S(M)) \quad , \quad S(M) = \Gamma(M) \times \Sigma(M), \quad (12)$$

where $U \subset M$ is an open subset, $\Gamma(M)$ is the $U(1)$ bundle and $\Sigma(M)$ is the spin bundle over M , respectively [27]. Standard computations show that the Dirac operator is an elliptic operator on M and since M is compact, the operator \mathcal{D} has a discrete spectrum of eigenvalues

$$\mathcal{D}\chi_n = \lambda_n \chi_n \quad , \quad n = 0, 1, 2, \dots \quad (13)$$

The spinor bundle $\Sigma(M)$ can be endowed with the following scalar product by which it becomes an Hilbert space

$$\langle \psi | \chi \rangle = \int d^4x \sqrt{g(x)} \psi^*(x) \chi(x). \quad (14)$$

Here, $*$ denotes the complex conjugation. The λ_n 's define a discrete family of functions on \mathcal{F} which is a consequence of the dependence of D on (e, A, ψ)

$$\lambda_n : \mathcal{F} \longrightarrow R \quad , \quad (e, A, \psi) \mapsto \lambda_n(e, A, \psi), \quad (15)$$

$$\lambda_n : \mathcal{F} \longrightarrow R^\infty \quad , \quad (e, A, \psi) \mapsto \{\lambda_n(e, A, \psi)\}, \quad (16)$$

where the second correspondence defines the infinite family of dynamical variables.

The functions (15) and (16) define a set of variables of the Euclidean supergravity. In principle, they could be found by solving the spectral problem (13) on arbitrary backgrounds which is a notorious difficult problem. Nevertheless, one can obtain interesting information about the eigenvalues λ_n 's by investigating their variation with respect to the variables (e, A, ψ) . In this way, one can check out explicitly the gauge invariance of the eigenvalues of the Dirac operator and one can attempt to construct their Poisson brackets. Note that the eigenvalues of the Dirac operator should be invariant under the transformations (7) since \mathcal{D} is covariant by construction.

3 On-Shell Poisson Brackets of $N = 2$ $D = 4$ Euclidean Supergravity

Let us study now the Poisson brackets on the covariant phase space of the on-shell Euclidean supergravity. A vector field on \mathcal{S} can be written as

$$X = \int d^4x \left(X_\mu^a(x)(e, A, \psi) \frac{\delta}{\delta e_\mu^a(x)} + X^\mu(x)(e, A, \psi) \frac{\delta}{\delta A^\mu(x)} + X_\mu^I(x)(e, A, \psi) \frac{\delta}{\delta \psi_\mu^I(x)} \right), \quad (17)$$

where $X_\mu^a(x)$, $X^\mu(x)$ and $X_\mu^I(x)$ are solutions of the equations of motion of $N = 2$ $D = 4$ supergravity. The vector field X defines a vector field $[X]$ on the covariant phase space \mathcal{G} as the equivalence class of (17) modulo the linearized gauge transformations. The symplectic two-form of the general relativity from [28] can be generalized to the supergravity by defining the following two-form field

$$\Omega(X, Y) = \int_N d^3\sigma \, n_\rho \left(X_\mu^a \overset{\rightrightarrows}{\nabla}_\sigma Y_\nu^b \epsilon_{ab\tau}^\sigma + X_\mu \overset{\rightrightarrows}{\nabla}_\tau Y_\nu + X_\mu^I \overset{\rightrightarrows}{\nabla}_\tau Y_\nu^J \epsilon_{IJ} \right) \epsilon^{\tau\rho\mu\nu}, \quad (18)$$

where N is a compact topological three-dimensional surface such that $M = N \times S^1$, n_ρ is its normal one-form and

$$F \overset{\rightrightarrows}{\nabla}_\sigma G = F \nabla_\sigma G - G \nabla_\sigma F. \quad (19)$$

Since the solutions of the Euclidean supergravity depends on the full supergravity multiplet, the covariant derivative that from (18) is the one that acts on the spinors (11). The components of Ω are defined in each sector by the following relations

$$\Omega_{ab}^{\mu\nu}(x, y) = \int_N d^3\sigma \, n_\rho \delta(x, \sigma(x)) \overset{\rightrightarrows}{\nabla}_\sigma \delta(y, x(\sigma)) \epsilon_{ab\tau}^\rho \epsilon^{\tau\sigma\mu\nu}, \quad (20)$$

$$\Omega^{\mu\nu}(x, y) = \int_N d^3\sigma \, n_\rho \delta(x, \sigma(x)) \overset{\rightrightarrows}{\nabla}_\sigma \delta(y, x(\sigma)) \epsilon^{\rho\sigma\mu\nu}, \quad (21)$$

$$\Omega_{IJ}^{\mu\nu}(x, y) = \int_N d^3\sigma \, n_\rho \delta(x, \sigma(x)) \overset{\rightrightarrows}{\nabla}_\sigma \delta(y, x(\sigma)) \epsilon_{IJ} \epsilon^{\rho\sigma\mu\nu}, \quad (22)$$

where $N : \sigma \rightarrow \xi(\sigma)$. Like the symplectic two-form of gravity which can be recognized in the first term of (18), Ω is degenerate in the gauge directions and has an inverse in the space of the orbits. Nevertheless, once the gauge fields and a representative of the vector field $[X]$ are fixed, then Ω admits an inverse Π . The components of Ω are defined by the following relations

$$\int d^4y \int d^4z \, \Pi_{\mu\nu}^{ab}(x, y) \Omega_{bc}^{\nu\rho}(y, z) X_\rho^c(z) = \int d^4z \, \delta(x, z) \delta_\mu^\rho \delta_c^a X_\rho^c(z), \quad (23)$$

$$\int d^4y \int d^4z \Pi_{\mu\nu}(x, y) \Omega^{\nu\rho}(y, z) X_\rho(z) = \int d^4z \delta(x, z) \delta_\mu^\rho X_\rho(z), \quad (24)$$

$$\int d^4y \int d^4z \Pi_{\mu\nu}^{IJ}(x, y) \Omega_{JK}^{\nu\rho}(y, z) X_\rho^K(z) = \int d^4z \delta(x, z) \delta_\mu^\rho \delta_K^I X_\rho^K(z), \quad (25)$$

where $\{X_\rho^c, X_\rho, X_\rho^K\}$ denote an arbitrary solution of the supergravity equations of motion which satisfies the chosen gauge. By using the relations (20), (21) and (22) in to (23), (24) and (25) one obtains a set of equations that defines the components of the inverse Π in the three sectors

$$\int_N d^3\sigma n_\rho \left(\Pi_{\mu\nu}^{ab}(x, x(\sigma)) \bar{\nabla}_\sigma X_\gamma^c(x(\sigma)) \right) \epsilon_{bc\tau}^\rho \epsilon^{\tau\nu\sigma\gamma} = X_\mu^a(x), \quad (26)$$

$$\int_N d^3\sigma n_\rho \left(\Pi_{\mu\nu}(x, x(\sigma)) \bar{\nabla}_\sigma X_\gamma(x(\sigma)) \right) \epsilon^{\rho\nu\sigma\gamma} = X_\mu(x), \quad (27)$$

$$\int_N d^3\sigma n_\rho \left(\Pi_{\mu\nu}^{IJ}(x, x(\sigma)) \bar{\nabla}_\sigma X_\gamma^K(x(\sigma)) \right) \epsilon_{JK} \epsilon^{\rho\nu\sigma\gamma} = X_\mu^I(x). \quad (28)$$

The Poisson brackets of two functions F, G on the space \mathcal{S} can be written as

$$\begin{aligned} \{F, G\} &= \int d^4x \int d^4y \left(\Pi_{\mu\nu}^{ab}(x, y) \frac{\delta F}{\delta e_\mu^a(x)} \frac{\delta G}{\delta e_\nu^b(y)} + \Pi_{\mu\nu}(x, y) \frac{\delta F}{\delta A_\mu(x)} \frac{\delta G}{\delta A_\nu(y)} \right. \\ &\quad \left. + \Pi_{\mu\nu}^{IJ}(x, y) \frac{\delta F}{\delta \psi_\mu^I(x)} \frac{\delta G}{\delta \psi_\nu^J(y)} \right). \end{aligned} \quad (29)$$

Note that the relations (26), (27) and (28) define the extension of the propagator of the linearized Einstein equations [7] to the linearized $N = 2$ $D = 4$ Euclidean supergravity in the chosen gauge. The Poisson brackets from (29) depend on the gauge, too. However, if F and G are gauge invariant functions then the Poisson brackets are gauge independent.

4 Poisson Brackets of the Dirac Eigenvalues

The local variation of the Dirac eigenvalues is given by the perturbation theory

$$\delta\lambda_n = \langle \chi_n | \delta\mathcal{D} | \chi_n \rangle. \quad (30)$$

The terms containing the local variation of the eigenspinors χ_n cancel each other. The variation of the Dirac eigenvalues under gauge transformations has the following form

$$\begin{aligned} \delta\lambda_n &= i \langle \chi_n | \gamma^a \delta e_a^\mu \nabla_\mu | \chi_n \rangle + \frac{i}{4} \langle \chi_n | \gamma^a e_a^\mu \delta \omega_\mu^{cd} \gamma_{cd} | \chi_n \rangle + g \langle \chi_n | \gamma^a e_a^\mu \delta A_\mu | \chi_n \rangle \\ &= i \langle \chi_n | \gamma^a \left(\xi^\nu \partial_\nu e_\mu^a + \partial_\mu \xi^\nu e_\nu^a - \xi_b^a e_\mu^b + \frac{1}{2} k \left(\psi_\mu^\dagger \gamma_5 \gamma^a \epsilon - \epsilon^\dagger \gamma_5 \gamma^a \psi_\mu \right) \right) \nabla^\mu | \chi_n \rangle \\ &\quad + \frac{i}{4} \langle \chi_n | \gamma^a e_a^\mu \delta \omega_\mu^{cd} \gamma_{cd} | \chi_n \rangle \\ &\quad + g \langle \chi_n | \gamma^a e_a^\mu \left(\xi^\nu \partial_\nu A_\mu + \partial_\mu \xi^\nu A_\nu + \partial_\mu \xi + \frac{i}{\sqrt{2}} \left(\epsilon^\dagger \gamma_5 \psi_\mu - \psi_\mu^\dagger \gamma_5 \epsilon \right) \right) | \chi_n \rangle, \end{aligned} \quad (31)$$

where the variation of the spin connection is given by the following relation

$$\delta\omega_\mu^{ab} = \delta\omega_\mu^{ab}(e) - \frac{k^2}{4} \left[\left(\delta e_\nu^a \psi^{\nu\dagger} + e_\nu^a \delta \psi^{\nu\dagger} \right) \gamma_5 \gamma_\mu \psi^b + \psi^{a\dagger} \gamma_5 \gamma_\mu \left(\delta e_\nu^b \psi^{\nu\dagger} + e_\nu^b \delta \psi^{\nu\dagger} \right) \right]$$

$$\begin{aligned}
& - \frac{k^2}{4} \left[\left(\delta e_\nu^a \psi^{\nu\dagger} + e_\nu^a \delta \psi^{\nu\dagger} \right) \gamma_5 \gamma^b \psi_\mu + \psi^{a\dagger} \gamma_5 \gamma^b \delta \psi_\mu \right] \\
& + \frac{k^2}{4} \left[\delta \psi_\mu^\dagger \gamma_5 \gamma^a \psi^b - \psi_\mu^\dagger \gamma_5 \gamma^a \left(\delta e_\nu^b \psi^\nu + e_\nu^b \delta \psi^\nu \right) \right] \\
& - (a \leftrightarrow b).
\end{aligned}$$

If the explicit form of the gauge transformations from (7) [26] is employed, one obtains the following variation of the Dirac eigenvalues

$$\begin{aligned}
\delta \lambda_n &= \delta \omega_\mu^{ab}(e) \\
& - \frac{k^2}{4} \left[\left(\xi^\rho \partial_\rho e_\nu^c + \partial_\nu \xi^\rho e_\rho^c - \xi_d^c e_\nu^d + \frac{k}{2} \left(\psi_\nu^\dagger \gamma_5 \gamma^c \epsilon - \epsilon^\dagger \gamma_5 \gamma^c \psi_\nu \right) \right) \left(\delta^{ca} \psi^{\nu\dagger} \gamma_5 \gamma_\mu \psi^b + \psi^{a\dagger} \gamma_5 \gamma_\mu \psi^{\nu\dagger} \delta^{cb} \right) \right. \\
& + \left(\xi^\rho \partial_\rho \psi^{\nu\dagger} + \partial^\nu \xi^\rho \psi^{\nu\dagger} - \frac{1}{4} \xi^{cd} \psi^{\nu\dagger} \gamma_{cd} + \frac{\nabla^\nu \xi^\dagger}{k} + \frac{i \xi^\dagger G^{\nu\rho} \gamma_\rho}{2\sqrt{2}} + \frac{g \xi^\dagger \gamma^\nu}{\sqrt{2}} - i g \xi \psi^{\nu\dagger} \right) e_\nu^a \gamma_5 \gamma_\mu \psi^b \\
& + \left. \psi^{a\dagger} \gamma_5 \gamma_\mu e_\nu^b \left(\xi^\rho \partial_\rho \psi^{\nu\dagger} + \partial^\nu \xi^\rho \psi^{\nu\dagger} - \frac{1}{4} \xi^{cd} \psi^{\nu\dagger} \gamma_{cd} + \frac{\nabla^\nu \xi^\dagger}{k} + \frac{i \xi^\dagger G^{\nu\rho} \gamma_\rho}{2\sqrt{2}} + \frac{g \xi^\dagger \gamma^\nu}{\sqrt{2}} - i g \xi \psi^{\nu\dagger} \right) \right] \\
& - \frac{k^2}{4} \left(\xi^\rho \partial_\rho e_\nu^a + \partial_\nu \xi^\rho e_\rho^a - \xi_d^a e_\nu^d + \frac{k}{2} \left(\psi_\nu^\dagger \gamma_5 \gamma^a \epsilon - \epsilon^\dagger \gamma_5 \gamma^a \psi_\nu \right) \right) \psi^{\nu\dagger} \gamma_5 \gamma^b \psi_\mu \\
& - \frac{k^2}{4} \left(\xi^\rho \partial_\rho \psi^{\nu\dagger} + \partial^\nu \xi^\rho \psi^{\nu\dagger} - \frac{1}{4} \xi^{cd} \psi^{\nu\dagger} \gamma_{cd} + \frac{\nabla^\nu \xi^\dagger}{k} + \frac{i \xi^\dagger G^{\nu\rho} \gamma_\rho}{2\sqrt{2}} + \frac{g \xi^\dagger \gamma^\nu}{\sqrt{2}} - i g \xi \psi^{\nu\dagger} \right) e_\nu^a \gamma_5 \gamma^b \psi_\mu \\
& - \frac{k^2}{4} \psi^{a\dagger} \gamma_5 \gamma^b \left(\xi^\nu \partial_\nu \psi_\mu + \partial_\mu \xi^\nu \psi_\nu + \frac{\xi^{cd} \gamma_{cd} \psi_\mu}{4} - \frac{\nabla_\mu \epsilon}{k} + \frac{i \gamma^\nu G_{\mu\nu} \epsilon}{2\sqrt{2}} - \frac{g \gamma_\mu \epsilon}{k^2 \sqrt{2}} + i g \xi \psi_\mu \right) \\
& + \frac{k^2}{4} \left(\xi^\rho \partial_\rho \psi_\mu^\dagger + \partial_\mu \xi^\rho \psi_\mu^\dagger - \frac{1}{4} \xi^{cd} \psi_\mu^\dagger \gamma_{cd} + \frac{\nabla_\mu \xi^\dagger}{k} + \frac{i \xi^\dagger G_m^{\rho} \gamma_\rho}{2\sqrt{2}} + \frac{g \xi^\dagger \gamma_\mu}{\sqrt{2}} - i g \xi \psi_\mu^\dagger \right) \gamma_5 \gamma^a \psi^b \\
& - \frac{k^2}{4} \psi_\mu^\dagger \gamma_5 \gamma^a \psi^\nu \left(\xi^\rho \partial_\rho e_\nu^b + \partial_\nu \xi^\rho e_\rho^b - \xi_d^b e_\nu^d + \frac{k}{2} \left(\psi_\nu^\dagger \gamma_5 \gamma^b \epsilon - \epsilon^\dagger \gamma_5 \gamma^b \psi_\nu \right) \right) \\
& - \frac{k^2}{4} \psi_\mu^\dagger \gamma_5 \gamma^a e_\rho^b \left(\xi^\nu \partial_\nu \psi^\rho + \partial^\rho \xi^\nu \psi_\nu + \frac{\xi^{cd} \gamma_{cd} \psi^\rho}{4} - \frac{\nabla^\rho \epsilon}{k} + \frac{i \gamma^\nu G_\nu^\rho \epsilon}{2\sqrt{2}} - \frac{g \gamma^\rho \epsilon}{k^2 \sqrt{2}} + i g \xi \psi^\rho \right) \\
& - (a \leftrightarrow b). \tag{32}
\end{aligned}$$

The above relation vanishes identically due to the gauge invariance of the theory but it can be used to check the explicit invariance of the Dirac eigenvalues. However, λ_n 's are not invariant under *arbitrary* variations of the supergravity multiplet.

Consider the linearized arbitrary transformations of the fields (e, A, ψ) with the real parameter v

$$e_{v\mu}^a = e_\mu^a + v \delta e_\mu^a, \tag{33}$$

$$A_{v\mu} = A_\mu + v \delta A_\mu, \tag{34}$$

$$\psi_{v\mu} = \psi_\mu + v \delta \psi_\mu. \tag{35}$$

The variation of the Dirac eigenvalues with respect to v can be computed from the following relation

$$\begin{aligned}
\left. \frac{d\lambda_n(e, A, \psi)}{dv} \right|_{v=0} &= \int d^4x \left[\frac{\delta \lambda_n(e, A, \psi)}{\delta e_\mu^a} \delta e_\mu^a + \frac{\delta \lambda_n(e, A, \psi)}{\delta A_\mu} \delta A_\mu + \frac{\delta \lambda_n(e, A, \psi)}{\delta \psi_\mu} \delta \psi_\mu \right] \\
&= \langle \chi_n | \left. \frac{d}{dv} \right|_{v=0} \mathcal{D} | \chi_n \rangle. \tag{36}
\end{aligned}$$

Using the following notations

$$T_{na}^\mu = \frac{\delta}{\delta e_\mu^a} S_n, \quad U_n^\mu = \frac{\delta}{\delta A_\mu} S_n, \quad V_n^{I\mu} = \frac{\delta}{\delta \psi_I^\mu} S_n, \quad (37)$$

$$S_n = \int d^4x \sqrt{\det(g_{\mu\nu})} (\chi_n^* \mathcal{D} \chi_n - \chi_n^* \chi_n). \quad (38)$$

one obtains the equations that give the variation of the eigenvalues λ_n 's under the general linear variation of the supergravity multiplet

$$\frac{d\lambda_n(e, A, \psi)}{\delta e_\mu^a} = T_{na}^\mu, \quad \frac{d\lambda_n(e, A, \psi)}{\delta A_\mu} = U_n^\mu, \quad \frac{d\lambda_n(e, A, \psi)}{\delta \psi_I^\mu} = V_n^{I\mu}. \quad (39)$$

The interpretation of the equations (39) is that they are the Jacobian matrix of the map (15). As in the case of general relativity, the above results allow one to study the map (15) in the space of supermultiplets by studying the quantities defined in (37). In principle, the quantities given by the above set of relations can be computed for any given supergravity background. With their help one can compute the Poisson brackets of the Dirac eigenvalues

$$\begin{aligned} \{\lambda_n, \lambda_m\} &= \int d^4x \int d^4y \left(T_{na}^\mu(x) \Pi_{\mu\nu}^{ab}(x, y) T_{mb}^\nu(y) + U_n^\mu(x) \Pi_{\mu\nu}(x, y) U_m^\nu(y) \right. \\ &\quad \left. + V_n^{I\mu}(x) \Pi_{\mu\nu}^{IJ}(x, y) V_m^{J\nu}(y) \right). \end{aligned} \quad (40)$$

The relation (40) represents the gauge independent Poisson brackets of two eigenvalues of the Dirac operator in terms of the objects computed in (39) and the propagator of the linearized equations of the $N = 2$ $D = 4$ on-shell supergravity.

Like in the case of the gravity, one can express the symplectic form in the covariant phase space \mathcal{G} in terms of the differentials of the Dirac eigenvalues

$$\Omega = \sum_{n,m} \Omega_{mn} d\lambda_n \wedge d\lambda_m, \quad (41)$$

where the coefficients Ω_{mn} should satisfy simultaneously the following set of relations

$$\sum_{m,n} \Omega_{mn} T_{na}^\mu(x) T_{mb}^\nu(y) = \Omega_{ab}^{\mu\nu}(x, y), \quad (42)$$

$$\sum_{m,n} \Omega_{mn} U_n^\mu(x) U_m^\nu(y) = \Omega^{\mu\nu}(x, y), \quad (43)$$

$$\sum_{m,n} \Omega_{mn} V_n^{I\mu}(x) V_m^{J\nu}(y) = \Omega_{\mu\nu}^{IJ}(x, y). \quad (44)$$

The relations (42), (43) and (44) hold only if the maps (15) and (16) are invertible on the phase space.

5 Conclusions

In this paperwe, we have shown that the Dirac eigenvalues form a set of dynamical variables of the on-shell $N = 2$ $D = 4$ Euclidean supergravity and that the notions of covariant phase space and the Poisson brackets can be extended to this system. In particular, the Poisson brackets can be computed for the Dirac eigenvalues by calculating the variation of λ_n 's under arbitrary transformations. The result is given in the relation (40) and it is gauge invariant.

The coefficients of the symplectic form Ω_{mn} can be expressed in terms of the coefficients on the phase space if the maps (15) and (16) are invertible. The similar question in general relativity was addressed in [20] and the answer was that there is a certain indeterminacy in finding the solutions of the equations that define the relation between the coefficients of the symplectic form in the two representations. It would be interesting to see if the supersymmetry and $U(1)$ symmetry can lift this indeterminacy. As in the case of the local $N = 1$ supersymmetry analysed in previous papers, the on-shell transformations define a set of maps from the space of smooth supermultiplets. These maps lead to a set of equations involving geometrical objects associated with the spacetime manifold which are interpreted as constraints on spacetime. When the maps are identified with the Dirac eigenvalues, the equations should vanish identically since λ^n 's are gauge invariant. Nevertheless, the constraints are still left behind and they relate the geometrical objects in spacetime. Another way to study the relationship between the Dirac eigenvalues and the geometry of the spacetime manifold is by using arbitrary transformations. Then a constraint is generated for any transformation for which $\delta\lambda_n$'s vanish

$$T_{na}^\mu \delta e_\mu^a + U_n^\mu \delta A_\mu + V_n^{I\mu} \delta \psi_\mu^I = 0. \quad (45)$$

However, it is not known, in general, whether such transformations exist and if they are compatible with the local supersymmetry.

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